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Twisted tori and fluxes: a no go theorem for Lie groups of weak G_2 holonomy[†]

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Abstract

In this paper we prove the theorem that there exists no 7–dimensional Lie group manifold $\mathcal{G}$ of weak G_2 holonomy. We actually prove a stronger statement, namely that there exists no 7–dimensional Lie group with negative definite Ricci tensor $\mathbf{Ric}_{IJ}$. This result rules out (supersymmetric and non–supersymmetric) Freund–Rubin solutions of M–theory of the form $\mathrm{AdS}_4 \times \mathcal{G}$ and compactifications with non–trivial 4–form fluxes of Englert type on an internal group manifold $\mathcal{G}$. A particular class of such backgrounds which, by our arguments are excluded as bulk supergravity compactifications corresponds to the so called compactifications on twisted–tori, for which $\mathcal{G}$ has structure constants $\tau^K{}_{IJ}$ with vanishing trace $\tau^J{}_{IJ} = 0$. On the other hand our result does not have bearing on warped compactifications of M–theory to four dimensions and/or to compactifications in the presence of localized sources (D–branes, orientifold planes and so forth). Henceforth our result singles out the latter compactifications as the preferred hunting grounds that need to be more systematically explored in relation with all compactification features involving twisted tori.

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1 Introduction

Recently considerable attention has been devoted to flux compactifications of super–string theory or of M–theory, since they provide mechanisms to stabilize the moduli fields [1]–[37]. Within this context a particularly interesting class of flux compactifications is represented by those on twisted–tori [1, 3, 38, 39, 11, 14, 25, 26, 28, 29, 31, 33, 34]. This is the contemporary understanding of the Scherk-Schwarz mechanism [1] of mass generation from extra dimensions. As it was explained by Hull and Reid-Edwards [27], twisted tori are just Lie group manifolds $\mathcal{G}$ modded by the action of some discrete subgroup $\Delta \subset \mathcal{G}$ which makes them compact.

A general pattern only recently elucidated is the relation between fluxes and gauge algebras. Upon dimensional reduction in the presence of fluxes one ends up with some lower dimensional gauged supergravity and a particularly relevant question is that about the structure of the gauge Lie algebra in relation with the choice of fluxes. Some years ago the authors of [40] introduced the concept of embedding matrix and by means of it classified all the $\mathcal{N} = 8$ supergravity gaugings, originally studied in [41, 42], were the electric group is taken to be $\mathrm{SL}(8, \mathbb{R}) \subset \mathrm{E}_{7(7)}$. The very idea of *embedding matrix* proved to be very fertile and pivotal, yet the classification of [40] was shown to be incomplete in [38, 39] by proving that the hypothesis that the electric group be $\mathrm{SL}(8, \mathbb{R})$ could be relaxed. In the same papers some new explicit examples of gaugings were also explicitly constructed. Later the authors of [43] obtained an elegant characterization of the embedding matrix as a suitable irreducible representation of $\mathrm{E}_{7(7)}$ thus extending the action of the global symmetry group of the ungauged theory (also referred to as U–duality group) to gauged supergravities. This analysis was eventually applied to gauged five–dimensional maximal supergravities in [44, 45]. Still later, from inspection of the low–energy description of various flux compactifications (either involving form–fluxes or "geometrical fluxes", i.e. background quantities related to the geometry of the internal manifold, like the "twist" tensor characterizing the twisted–tori), a precise statement about the relation between internal fluxes and local symmetries of the lower–dimensional effective supergravity was derived: the background quantities enter the lower–dimensional gauged supergravity as components of the embedding tensor, and thus they can be naturally assigned to representations of the U–duality group [26] (a similar statement was made in the context of the heterotic string in [3]). Hence an obvious question is that relative to the gauge algebra emerging from flux compactifications on twisted tori. This question has been addressed in a recent series of papers [25, 28, 29, 31] and it has been advocated that the gauge algebraic structures emerging in $D = 4$ supergravities which originate from this kind of M–theory compactifications, do not fall in the class of Lie algebras $\mathbb{G}$, but rather have to be understood in the more general context of Free Differential Algebras. The algebraic structure that goes under this name was independently discovered at the beginning of the eighties in Mathematics by Sullivan [46] and in Physics by the authors of [47]. Free Differential Algebras (FDA) are a categorical extension of the notion of Lie algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theory, hence also of string theory. The reason is the ubiquitous presence in the spectrum of string/supergravity theory of antisymmetric gauge fields (p–forms) of rank greater than one. The very existence of

FDA.s is a consequence of Chevalley cohomology of ordinary Lie algebras and Sullivan has provided us with a very elegant classification scheme of these algebras based on two structural theorems rooted in the set up of such an elliptic complex.

In view of Sullivan's theorems one of the present authors analyzed in [48] the results of [25, 28, 29, 31] from the point of view of Chevalley cohomology. The goal was that of establishing the structure of the minimal FDA algebra $\mathbb{M}$ which emerges from twisted tori compactifications and relating its *generalized structure constants* to the fluxes, just in the spirit of the general relation between *gaugings*, i.e. embedding matrices and p-form fluxes explained above. The basic notion of *minimal algebra* is illustrated in the following way.

As it was noted in [49], FDA.s have the additional fascinating property that, as opposite to ordinary Lie algebras, they already encompass their own gauging. Indeed the first of Sullivan's structural theorems, which is in some sense analogous to Levi's theorem for Lie algebras, states that the most general FDA is a semidirect sum of a so called minimal algebra $\mathbb{M}$ with a contractible one $\mathbb{C}$. The generators of the minimal algebra are physically interpreted as the connections or *potentials*, while the contractible generators are physically interpreted as the *curvatures*. The real hard–core of the FDA is the minimal algebra and it is obtained by setting the contractible generators (the curvatures) to zero. The structure of the minimal algebra $\mathbb{M}$, in turn, is beautifully determined by Chevalley cohomology of the underlying Lie algebra $\mathbb{G}$. This happens to be the content of Sullivan's second structural theorem.

By rephrasing all the equations of papers [25, 28, 29, 31] in the framework of Chevalley cohomology and exploiting the underlying structure of a double elliptic complex, paper [48] established the following result:

Theorem 1.1 *The minimal algebra $\mathbb{M}$ emerging in twisted tori compactification of M-theory, i.e. in compactifications on a 7-dimensional group manifold $\mathcal{G}$ necessarily coincides with the Lie algebra $\mathbb{G}_7$ of $\mathcal{G}$ unless the internal flux g_{IJKL} of the 4-form defines a cohomologically non-trivial 4-cycle of $\mathbb{G}_7$, namely unless $g_{IJKL}e^I \wedge e^J \wedge e^K \wedge e^L \equiv \Delta^{(0,4)} \in \mathrm{H}^{(4)}(\mathbb{G}_7)$.*

It was actually proven in [48] that in order to get a minimal algebra which is a proper extension of $\mathbb{G}_7$ on a certain background, $\Delta^{(0,4)} \in \mathrm{H}^{(4)}(\mathbb{G}_7)$ is just a necessary but not yet sufficient condition. Indeed it was proven the following

Theorem 1.2 *The necessary and sufficient condition for the minimal free differential algebra $\mathbb{M}$ to be a proper extension of $\mathbb{G}_7$, is that the 4-form $\Delta^{(0,4)}$ defined by the internal 4–form flux should:*

a *be cohomologically non trivial $\Delta^{(0,4)} \in \mathrm{H}^{(4)}(\mathbb{G}_7)$*

b *its triple contraction should be a non trivial 1–cycle. Given a basis of cycles $\Gamma^{[p]}$ for each Chevalley cohomology group $H^{(p)}(\mathbb{G}_7)$, and the pairing form $<,>$, this condition is expressed by:*

$$\exists\, \Gamma^{[6]}_{\alpha} \in \mathrm{H}^{(6)}(\mathbb{G}_7) \quad \backslash \quad 0 \neq \langle i_W \circ i_W \circ i_W \, \Delta^{[0,4]} \, , \, \Gamma^{[6]}_{\alpha} \rangle \tag{1.1}$$

Hence paper [48] reduced the problem of establishing whether or not the minimal part of the FDA.s, that emerge from twisted tori compactifications of M–theory, involve forms other than the Kaluza–Klein vectors W^I_μ, to the following three questions about 7-dimensional Lie algebras $\mathbb{G}_7$:

A Do 7-dimensional Lie algebras $\mathbb{G}_7$ exist which admit a constant rank 4 antisymmetric tensor $g_{IJKL} \in \bigwedge^{(4)} \mathrm{adj}\mathbb{G}_7$ which together with the Ricci 2-form of the same algebra $\mathbf{Ric}(\mathbb{G}_7)$ provides an exact solution of M–theory field equations?

B If the answer to question [A] is yes, can $\Delta^{[0,4]} \equiv g_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L$ be chosen cohomologically non trivial, namely within $\mathrm{H}^{(4)}(\mathbb{G}_7)$?

C If the answer to questions [A] and [B] is yes, can the 4-cycle $\Delta^{[0,4]}$ be chosen in such a way that it also satisfies the condition (1.1)?

In the present paper by using general classical theorems in Lie algebra theory we shall prove that the answer to question [A] is already negative so that questions [B] and [C] need not even be addressed. This is the no-go theorem announced by our title. Let us however immediately comment on the scope and bearings of our theorem. Indeed, we are quite conscious that the value of any negative result resides in a clear–cut presentation of the hypotheses and conditions under which it is derived. Only in this case the negative statement can be turned into a positive contribution by indicating the directions to be pursued in order to evade it.

So let us immediately recall that by compactification of M-theory on a twisted torus we mean in this paper the same that was meant in [48] and that was assumed in the series of papers [25, 28, 29, 31], namely a bosonic configuration of the M–theory light fields (the metric $g_{\hat{\mu}\hat{\nu}}$ and the three form $A_{\hat{\mu}\hat{\nu}\hat{\rho}}$) such that the 11–dimensional manifold splits into the following direct product:

$$\mathcal{M}_{11} = \mathcal{M}_4 \times \mathcal{G}/\Delta \tag{1.2}$$

$\mathcal{M}_4$ denoting a four-dimensional maximally symmetric manifold whose coordinates we denote x^μ and $\mathcal{G}$ a 7–dimensional group manifold whose parameters we denote y^I. Let us denote by

$$e^I = e^I_J(y)\, dy^J \tag{1.3}$$

the purely y-dependent left–invariant 1-forms on $\mathcal{G}$ which, by definition, satisfy the Maurer–Cartan equations:

$$\partial e^I = \tfrac{1}{2} \tau^I{}_{JK}\, e^J \wedge e^K \;\; ; \;\; I, J, K = 4, \dots\ 10 \tag{1.4}$$

$\tau^I{}_{JK}$ being the structure constants of the Lie algebra $\mathbb{G}_7$:

$$[T_I\,, T_J] \;=\; \tau^K_{\ IJ}\, T_K \tag{1.5}$$

It is a constitutive part of what we understand by compactification that in any configuration of the compactified theory the eleven dimensional vielbein is split as follows:

$$V^{\hat{a}} = \left\{ \begin{array}{rclcrcl} V^r &=& E^r(x) & ; & r &=& 0,1,2,3 \\ V^a &=& \Phi^a{}_J(x)\, \left(e^I + W^I(x)\right) & ; & a &=& 4,5,6,7,8,9,10 \end{array} \right. \tag{1.6}$$

where $E^r(x)$ is a purely x–dependent 4–dimensional vielbein, $W^I(x)$ is an x–dependent 1–form on x-space describing the Kaluza Klein vectors and the purely x–dependent 7×7 matrix $\Phi^a{}_J(x)$ encodes part of the scalar fields of the compactified theory, namely the internal metric moduli. At the same time the 3-form is expanded as follows:

$$\mathbf{A}^{[3]} = C^{[0]}_{IJK}(x)\, V^I \wedge V^J \wedge V^K + A^{[1]}_{IJ}(x) \wedge V^I \wedge V^J + B^{[2]}_I(x) \wedge V^I + A^{[3]}(x) \tag{1.7}$$

where $V^I = e^I + W^I(x)$, $C^{[0]}_{IJK}(x)$ are x-dependent scalar fields, $A^{[1]}_{IJ}(x)$ are x-dependent 1-forms $B^{[2]}_I(x)$ are x-dependent 2-forms and finally $A^{[3]}(x)$ is an x–dependent 3-form. All of these forms are assumed to live on $D = 4$ space-time. From these assumptions it follows that the bosonic field strength is expanded as follows:

$$\begin{aligned} \mathbf{F}^{[4]} \equiv\; & F^{[4]}(x) + F^{[3]}_I(x) \wedge V^I + F^{[2]}_{IJ}(x) \wedge V^I \wedge V^J \\ & + F^{[1]}_{IJK}(x) \wedge V^I \wedge V^J \wedge V^K + F^{[0]}_{IJKL}(x) \wedge V^I \wedge V^J \wedge V^K \wedge V^L \end{aligned} \tag{1.8}$$

where $F^{[p]}_{I_1 \dots I_{4-p}}(x)$ are x-space p–forms depending only on x.

In bosonic backgrounds with space–time geometry (1.2), the family of configurations (1.6) must satisfy the condition that by choosing:

$$E^r = \text{vielbein of a maximally symmetric 4-dimensional space time} \tag{1.9}$$
$$\Phi^I{}_J(x) = \delta^I{}_J \tag{1.10}$$
$$W^I = 0 \tag{1.11}$$
$$F^{[3]}_I(x) = F^{[2]}_{IJ}(x) = F^{[1]}_{IJK}(x) = 0 \tag{1.12}$$
$$F^{[4]}(x) = e\, \epsilon_{rstu}\, E^r \wedge E^s \wedge E^t \wedge E^u \quad ; \quad (e = \text{constant parameter}) \tag{1.13}$$
$$F^{[0]}_{IJKL}(x) = g_{IJKL} = \text{constant tensor} \tag{1.14}$$

we obtain an exact *bona fide* solution of the eleven–dimensional field equations of M-theory.

There are three possible eleven–dimensional backgrounds of this kind:

$$\mathcal{M}_4 = \begin{cases} \mathcal{M}_4 & \text{Minkowsky space} \\ \mathrm{dS}_4 & \text{de Sitter space} \\ \mathrm{AdS}_4 & \text{anti de Sitter space} \end{cases} \tag{1.15}$$

In any case Lorentz invariance imposes eqs.(1.10,1.11,1.12) while translation invariance imposes that the vacuum expectation value of the scalar fields $\Phi^I{}_J(x)$ should be a constant matrix

$$< \Phi^I{}_J(x) >= \mathcal{A}^I_J \tag{1.16}$$

while the tensor g_{IJKL} should be a constant tensor as assumed in eq.(1.14). The reason why we set $\mathcal{A}^I_J = \delta^I{}_J$ is that any matrix $\mathcal{A}$ can be reabsorbed by a change of basis of generators of the Lie algebra and hence it corresponds to no new degrees of freedom.

In this paper we shall prove that there are no $\mathbb{G}_7$ Lie algebras such that eq.s (1.9-1.14) lead to a bona fide solution of M-theory field equations with a cohomologically non–trivial

flux g_{IJKL}. This, in view of our previous discussion, implies the no go theorem on FDA claimed above. The same result was obtained in [31, 36] from inspection of the scalar potential $\mathcal{V}$ of the dimensionally reduced theory. Indeed bosonic backgrounds of the kind we are considering, correspond to extrema of $\mathcal{V}$. It was shown that the extremization of $\mathcal{V}$ with resect to the axions C_{IJK} implies the vanishing of the internal components $F^{[0]}_{IJKL}$ of the 4–form field strength:

$$F^{[0]}_{IJKL} = -g_{IJKL} - \frac{3}{2}\tau^M{}_{[IJ}C_{KL]M} = 0 \qquad (1.17)$$

which in turn implies that g_{IJKL} is cohomologically trivial. Still from inspection of $\mathcal{V}$ in [36], solutions with de Sitter geometry ($\mathcal{V} > 0$ at the minimum) were ruled out, but no statement was made about the existence of anti–de Sitter vacua. In [33] the existence of supersymmetric anti–de Sitter vacua was ruled out in a particular orientifold truncation of the maximally supersymmetric model. The output of our present analysis is two–fold: on one hand we give an alternative formal derivation of the triviality of g_{IJKL}, making use of the aforementioned arguments, on the other hand we show that bosonic backgrounds of anti–de Sitter type (arising from Freund –Rubin type of compactifications) are not solutions of the theory.

It should be noted that there are two main assumptions which could be possibly relaxed leading, may be, to different conclusions:

1. We could introduce a warp factor. This means that in eq. (1.6) the first line could be replaced by
$$V^a = \exp[\mathcal{W}(y)]\, E^a(x) \qquad (1.18)$$
where $\mathcal{W}(y)$ is some suitable function of the internal coordinates. In this case, however, the entire analysis of eq.s (1.7) and (1.8) in terms of Chevalley cohomology has to be reconsidered since the internal boundary operator ∂, defining such a cohomology, no longer annihilates the $D = 4$ vielbeins ($\partial V^a \neq 0$).

2. Our result applies to the field equations of M-theory without sources. Including the contribution of $M2$ or $M5$ branes one changes those equations by the addition of new terms whose bearing has to be considered.

The token by means of which we shall prove our negative result is provided by the reduction of the whole problem to a question of holonomy on the considered internal manifold $\mathcal{M}_7$ (in our case the group manifold $\mathcal{G}$). Indeed as we explain in detail in section 2 a solution of M-theory field equations with a non–trivial internal flux does exist if and only if the internal 7-manifold $\mathcal{M}_7$ is a Riemanian manifold of *weak* G_2 *holonomy*[50]. This is the new phrasing, in modern parlance, of the notion originally introduced at the beginning of the eighties by the authors of [51] under the name of *Englert manifolds*.

In view of this the final question will be whether 7-dimensional group manifolds of *weak* G_2 *holonomy* do or do not exist. Compact coset manifolds $\mathcal{S}/\mathcal{R}$ of the Englert type (weak G_2 holonomy) were all classified and studied in the golden age of Kaluza Klein supergravity namely at the beginning of the eighties[1]. Yet group manifolds, possibly

[1]See [53] for a comprehensive review and the basic original literature for the various families of $\mathcal{S}/\mathcal{R}$ spaces namely [54, 55, 56] for $\mathbb{S}^7$, [57] for the squashed $\mathbb{S}^7$, [58] for the M^{pqr} spaces, [59] for Q^{pqr} spaces, [60] for the N^{pqr} spaces and [61] for the exhaustive classification of the remaining ones.

non–compact, have not been considered so far under this point of view. We shall be able to prove the:

Theorem 1.3 *7-dimensional group manifolds of Englert type, namely of weak* G_2 *holonomy do not exist.*

Using the interesting results of [50] we will reformulate the condition of weak G_2 holonomy in terms of the irreducible G_2 representation contents of the candidate structure constants $\tau^I{}_{JK}$ of the Lie algebra $\mathbb{G}_7$. This leads to a parametrization of $\tau^I{}_{JK}$ in terms of 91–parameters. We should now impose Jacobi indentities and find how many solutions survive. This turns out to be algebraically too difficult. Yet we can use a different powerful argument. If a Lie algebra of weak G_2 holonomy existed its Ricci tensor $\mathbf{Ric}^I{}_J$[2] would be proportional to the identity matrix with a *positive* coefficient. We can instead prove that for all possible 7-dimensional Lie algebras the Ricci form $\mathbf{Ric}^I{}_J$ has an indefinite signature admitting at least one negative or null eigenvalue.

Our paper is organized as follows

In section 2 we analyze *M*-theory field equations and the notion of weak G_2 holonomy. We show that this is identical to the notion of *Englert manifold* introduced in the literature on Kaluza Klein supergravity two decades ago and we demonstrate that this is the necessary and sufficient condition in order to find solutions with an internal flux.

In section 3 we reformulate the notion of weak G_2 holonomy in terms of representation theory.

In section 4 we prove our main no-go theorem.

Section 5 briefly summarizes our conclusions.

Appendix A contains several details on G_2 invariant forms, projection operators and irreducible tensors.

Appendix B contains a classification of seven dimensional algebras with either SO(3) or $\mathrm{SO}(1,2)$ Levi subalgebras. These classifications are utilized in the proof of our main no-go theorem.

2 M-theory field equations and 7-manifolds of weak G_2 holonomy *i.e.* Englert 7-manifolds

In the recent literature about flux compactifications a geometrical notion which has been extensively exploited is that of G-structures.

Following, for instance, the presentation of [62], if $\mathcal{M}_n$ is a differentiable manifold of dimension n, $T\mathcal{M}_n \xrightarrow{\pi} \mathcal{M}_n$ its tangent bundle and $F\mathcal{M}_n \xrightarrow{\pi} \mathcal{M}_n$ its frame bundle, we say that $\mathcal{M}_n$ admits a G-structure when the structural group of $F\mathcal{M}_n$ is reduced from the generic $\mathrm{GL}(\mathrm{n},\mathbb{R})$ to a proper subgroup $\mathrm{G} \subset \mathrm{GL}(\mathrm{n},\mathbb{R})$. Generically, tensors on $\mathcal{M}_n$ transform in representations of the structural group $\mathrm{GL}(\mathrm{n},\mathbb{R})$. If a G-structure reduces

[2]In our conventions the curvature tensor is $-1/2$ times the definition which is traditionally adopted in the literature on General Relativity, so that for instance the Ricci tensor $\mathbf{Ric}_{IJ}$ of a sphere is negative definite. Moreover we use the "mostly minus" convention for the metric, so that the internal manifold has a negative definite metric.

this latter to $G \subset GL(n,\mathbb{R})$, then the decomposition of an irreducible representation of $GL(n,\mathbb{R})$, pertaining to a certain tensor t^p, with respect to the subgroup G may contain singlets. This means that on such a manifold $\mathcal{M}_n$ there may exist a certain tensor t^p which is G–invariant, and therefore globally defined. As recalled in [62] existence of a Riemannian metric g on $\mathcal{M}_n$ is equivalent to a reduction of the structural group $GL(n,\mathbb{R})$ to O(n), namely to an O(n)-structure. Indeed, one can reduce the frame bundle by introducing orthonormal frames, the vielbein e^I, and, written in these frames, the metric is the the O(n) invariant tensor δ_{IJ}. Similarly orientability corresponds to an SO(n)-structure and the existence of spinors on spin manifolds corresponds to a Spin(n)-structure.

In the case of seven dimensions, an orientable Riemannian manifold $\mathcal{M}_7$, whose frame bundle has generically an SO(7) structural group admits a G_2-structure if and only if, in the basis provided by the orthonormal frames e^I, there exists an antisymmetric 3-tensor ϕ_{IJK} satisfying the algebra of the octonionic structure constants:

$$\begin{aligned} \phi_{ABK}\,\phi_{CDK} &= \tfrac{1}{18}\,\delta^{CD}_{AB} - \tfrac{2}{3}\,\phi^{\star}_{ABCD} \\ -\tfrac{1}{6}\,\epsilon_{IJKABCD}\,\phi^{\star}_{ABCD} &= \phi_{IJK} \end{aligned} \tag{2.1}$$

which is invariant, namely it is the same in all local trivializations of the SO(7) frame bundle. This corresponds to the algebraic definition of G_2 as that subgroup of SO(7) which acts as an automorphism group of the octonion algebra. Alternatively G_2 can be defined as the stability subgroup of the 8-dimensional spinor representation of SO(7). Hence we can equivalently state that a manifold $\mathcal{M}_7$ has a G_2-structure if there exists at least an invariant spinor η, which is the same in all local trivializations of the Spin(7) spinor bundle.

In terms of this invariant spinor the invariant 3–tensor ϕ_{IJK} has the form [3]:

$$\phi_{IJK} = \tfrac{1}{6}\,\eta^T\,\gamma_{IJK}\,\eta \tag{2.2}$$

and eq.(2.2) provides the relation between the two definitions of the G_2-structure.

On the other hand the manifold has not only a G_2–structure, but also G_2–holonomy if the invariant three–tensor ϕ_{ABK} is covariantly constant. Namely we must have:

$$0 \;=\; \nabla\phi_{ABK} \,\equiv\, d\phi_{ABK} \,+\, 3\,\omega_{P[I}\,\phi_{JK]P} \tag{2.3}$$

where the 1-form ω^{AB} is the spin connection of $\mathcal{M}_7$. Alternatively the manifold has G_2–holonomy if the invariant spinor η is covariantly constant, namely if:

$$\exists\,\eta \,\in\, \Gamma(\mathrm{Spin}\mathcal{M}_7,\mathcal{M}_7) \quad \backslash \quad 0 \,=\, \nabla\,\eta \,\equiv\, d\eta - \tfrac{1}{4}\,\omega^{IJ}\,\gamma_{IJ}\,\eta \tag{2.4}$$

where γ_I ($I = 1,\ldots,7$) are the 8×8 gamma matrices of the SO(7) Clifford algebra. The relation between the two definitions (2.3) and (2.4) of G_2-holonomy is the same as for the two definitions of the G_2-structure, namely it is given by eq.(2.2). As a consequence of its own definition a Riemannian 7-manifold with G_2 holonomy is Ricci flat. Indeed the integrability condition of eq.(2.4) yields:

$$\mathcal{R}^{AB}_{JK}\,\gamma_{AB}\,\eta \,=\, 0 \tag{2.5}$$

[3]For more details on the G_2-formalism see appendix A

where $\mathcal{R}^{AB}_{JK}$ is the Riemann tensor of $\mathcal{M}_7$. From eq.(2.5), by means of a few simple algebraic manipulations one obtains two results:

- The curvature 2-form

$$\mathcal{R}^{AB} \equiv \mathcal{R}^{AB}_{JK}\, e^J \wedge e^K \tag{2.6}$$

  is G_2 Lie algebra valued, namely it satisfies the condition[4]:

$$\phi^{IAB}\,\mathcal{R}^{AB} = 0 \tag{2.7}$$

  which projects out the **7** of G_2 from the **21** of SO(7) and leaves with the adjoint **14**.

- The internal Ricci tensor is zero:

$$\mathcal{R}^{AM}_{BM} = 0 \tag{2.8}$$

Next we consider the field equations of M-theory [63], which we write here with the notations and the conventions of [64]

$$0 = \mathcal{D}_{\hat{m}}\, F^{\hat{m}\hat{c}_1\hat{c}_2\hat{c}_3} + \frac{1}{96}\,\epsilon^{\hat{c}_1\hat{c}_2\hat{c}_3\hat{a}_1\ldots\hat{a}_8}\, F_{\hat{a}_1\ldots\hat{a}_4}\, F_{\hat{a}_5\ldots\hat{a}_8} \tag{2.9}$$

$$R^{\hat{a}\hat{m}}{}_{\hat{b}\hat{m}} = 6F^{\hat{a}\hat{c}_1\hat{c}_2\hat{c}_3}\, F_{\hat{b}\hat{c}_1\hat{c}_2\hat{c}_3} - \tfrac{1}{2}\delta^{\hat{a}}_{\hat{b}}\, F^{\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4}\, F_{\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4} \tag{2.10}$$

where hatted indices run on eleven values and are flat indices. We make the compactification ansatz:

$$\mathcal{M}_{11} = \mathcal{M}_7 \times \mathcal{M}_4 \tag{2.11}$$

where $\mathcal{M}_4$ is one of the three possibilities mentioned in eq.(1.15) and all of eq.s(1.10-1.14) hold true. Then we split the rigid index range as follows:

$$\hat{a}, \hat{b}, \hat{c}, \ldots = \begin{cases} a, b, c, \ldots = 4, 5, 6, 7, 8, 9, 10 = \mathcal{M}_7 \text{ indices} \\ r, s, t, \ldots = 0, 1, 2, 3 = \mathcal{M}_4 \text{ indices} \end{cases} \tag{2.12}$$

and by following the conventions employed in [58] and using the results obtained in the same paper, we conclude that the compactification ansatz reduces the system (2.9,2.10) to the following one:

$$R^{rs}{}_{tu} = \lambda\, \delta^{rs}_{tu} \tag{2.13}$$

$$\mathcal{R}^{IK}_{JK} = 3\,\nu\, \delta^I_J \tag{2.14}$$

$$F_{rstu} = e\, \epsilon_{rstu} \tag{2.15}$$

$$g_{IJKL} = f\, \mathcal{F}_{IJKL} \tag{2.16}$$

$$\mathcal{F}^{AIJK}\, \mathcal{F}_{BIJK} = \mu\, \delta^A_B \tag{2.17}$$

$$\mathcal{D}^M\, \mathcal{F}_{MIJK} = \tfrac{1}{2}\, e\, \epsilon_{IJKPQRS}\, \mathcal{F}^{PQRS} \tag{2.18}$$

recall that we use capital latin indices to label the internal coordinates. Eq. (2.14) states that the internal manifold $\mathcal{M}_7$ must be an Einstein space. Eq.s (2.15) and (2.16)

[4] See appendix A for details on G_2 decomposition of SO(7) representations

state that there is a flux of the four–form both on 4–dimensional space-time $\mathcal{M}_4$ and on the internal manifold $\mathcal{M}_7$. The parameter e, which fixes the size of the flux on the four–dimensional space and was already introduced in eq.(1.13), is called the Freund-Rubin parameter [54]. As we are going to show, in the case that a non vanishing $\mathcal{F}^{AIJK}$ is required to exist, eq.s (2.17) and (2.18), are equivalent to the assertion that the manifold $\mathcal{M}_7$ has weak G_2 holonomy rather than G_2–holonomy, to state it in modern parlance [50]. In paper [51], manifolds admitting such a structure were instead named *Englert spaces* and the underlying notion of weak G_2 holonomy was already introduced there with the different name of *de Sitter* $\mathrm{SO}(7)^+$ *holonomy.*

Indeed eq.(2.18) which, in the language of the early eighties was named Englert equation [65] and which is nothing else but equation (2.9), upon substitution of the Freund Rubin ansatz (2.15) for the external flux, can be recast in the following more revealing form: Let

$$\Phi^\star \equiv \mathcal{F}_{IJKL}\, e^I \wedge e^J \wedge \epsilon^K \wedge e^L \tag{2.19}$$

be a the constant 4–form on $\mathcal{M}_7$ defined by our non vanishing flux, and let

$$\Phi \equiv \tfrac{1}{24}\epsilon_{ABCIJKL}\,\mathcal{F}_{IJKL}\, e^A \wedge e^B \wedge \epsilon^C \tag{2.20}$$

be its dual. Englert eq.(2.18) is just the same as writing:

$$\begin{aligned} d\Phi &= 12\, e\, \Phi^\star \\ d\Phi^\star &= 0 \end{aligned} \tag{2.21}$$

When the Freund Rubin parameter vanishes $e = 0$ we recognize in eq.(2.21) the statement that our internal manifold $\mathcal{M}_7$ has G_2-holonomy and hence it is Ricci flat. Indeed Φ is the G_2 invariant and covariantly constant form defining G_2-structure and G_2-holonomy. On the other hand the case $e \neq 0$ corresponds to the weak G_2 holonomy. Just as we reduced the existence of a closed three-form Φ to the existence of a G_2 covariantly constant spinor satisfying eq.(2.4) which allows to set the identification (2.2), in the same way eq.s (2.21) can be solved *if and only if* on $\mathcal{M}_7$ there exist a weak Killing spinor η satisfying the following defining condition:

$$\mathcal{D}_I\, \eta = m\, e\, \gamma_I\, \eta \tag{2.22}$$

where m is a numerical constant and e is the Freund-Rubin parameter, namely the only scale which at the end of the day will occur in the solution.

The integrability of the above equation implies that the Ricci tensor be proportional to the identity, namely that the manifold is an Einstein manifold and furthermore fixes the proportionality constant:

$$\mathcal{R}^{IM}{}_{JM} = 12\, m^2\, e^2\, \delta^I_J \quad \longrightarrow \nu = 12\, m^2\, e^2 \tag{2.23}$$

In case such a spinor exist, by setting:

$$g_{IJKL} = \mathcal{F}_{IJKL} = \eta^T\, \gamma_{IJKL}\eta = 24\, \phi^\star_{IJKL} \tag{2.24}$$

we find that Englert equation (2.18) is satisfied, provided we have:

$$m = -\frac{3}{2} \tag{2.25}$$

In this way Maxwell equation, namely (2.9) is solved. Let us also note, as the authors of [51] did many years ago, that condition (2.22) can also be interpreted in the following way. The spin-connection ω^{AB} plus the vielbein e^C define on any non Ricci flat 7-manifold $\mathcal{M}_7$ a connection which is actually SO(8) rather than SO(7) Lie algebra valued. In other words we have a principal SO(8) bundle which leads to an SO(8) spin bundle of which η is a covariantly constant section:

$$0 = \nabla^{\mathrm{SO}(8)}\eta = \nabla^{\mathrm{SO}(7)} - m\, e\, e^I\, \gamma_I\, \eta \tag{2.26}$$

The existence of η implies a reduction of the SO(8)-bundle. Indeed the stability subgroup of an SO(8) spinor is a well known subgroup $\mathrm{SO}(7)^+$ different from the standard SO(7) which, instead, stabilizes the vector representation. Hence the so named weak G_2 holonomy of the SO(7) spin connection ω^{AB} is the same thing as the $\mathrm{SO}(7)^+$ holonomy of the SO(8) Lie algebra valued *de Sitter connection* $\{\omega^{AB}, e^C\}$ introduced in [51] and normally discussed in the old literature on Kaluza Klein Supergravity.

We have solved Maxwell equation, but we still have to solve Einstein equation, namely (2.10). To this effect we note that:

$$\mathcal{F}_{BIJK}\, \mathcal{F}^{AIJK} = 24\, \delta^A_B \quad \Longrightarrow \quad \mu = 24 \tag{2.27}$$

and we observe that eq.(2.10) reduces to the following two conditions on the parameters (see [58] for details) :

$$\begin{aligned} \tfrac{3}{2}\,\lambda &= -\left(24\, e^2 + \tfrac{7}{2}\,\mu\, f^2\right) \\ 3\,\nu &= 12\, e^2 + \tfrac{5}{2}\,\mu\, f^2 \end{aligned} \tag{2.28}$$

From eq.s (2.28) we conclude that there are only three possible kind of solutions.

a The flat solutions of type

$$\mathcal{M}_{11} = Mink_4 \otimes \underbrace{\mathcal{M}_7}_{\text{Ricci flat}} \tag{2.29}$$

where both $D = 4$ space-time and the internal 7-space are Ricci flat. These compactifications correspond to $e = 0$ and $F_{IJKL} = 0 \Rightarrow g_{IJKL} = 0$. In this category fall the typical twisted tori compactifications which implement the Scherk and Schwarz mechanism but do not support any internal flux. So, in view of the results of [48], they cannot lead to any FDA in $D = 4$ with minimal subalgebra larger than $\mathbb{G}_7$.

b The Freund Rubin solutions of type

$$\mathcal{M}_{11} = \mathrm{AdS}_4 \otimes \underbrace{\mathcal{M}_7}_{\text{Einst. manif.}} \tag{2.30}$$

These correspond to anti de Sitter space in 4-dimensions, whose radius is fixed by the Freund Rubin parameter $e \neq 0$ times any Einstein manifold in 7–dimensions with no internal flux, namely $g_{IJKL} = 0$. Once again, in view of [48], also these compactifications lead to FDAs in $D = 4$ which have $\mathbb{G}_7$ as the minimal part.

c The Englert type solutions

$$\mathcal{M}_{11} \;=\; \mathrm{AdS}_4 \otimes \underbrace{\mathcal{M}_7}_{\substack{\text{Einst. manif.}\\ \text{weak } \mathrm{G}_2 \text{ hol}}} \tag{2.31}$$

These correspond to anti de Sitter space in 4-dimensions ($e \neq 0$) times a 7–dimensional Einstein manifold which is necessarily of weak G_2 holonomy in order to support a consistent non vanishing internal flux g_{IJKL}. These are the only possible candidate compactifications for the generation of FDA in $D = 4$ in which the minimal part is a proper extension of $\mathbb{G}_7$. In the sequel we concentrate on Englert solutions.

In view of the previous discussion we set $\mathcal{F}_{IJKL} \neq 0$ and we complete the analysis of our parameter equations, just recalling the results presented in [58]. Combining eq.s (2.28) with the previous ones we uniquely obtain:

$$\lambda = -30\, e^2 \quad ; \quad f = \pm \tfrac{1}{2}\, e \tag{2.32}$$

Summarizing, provided, the manifold $\mathcal{M}_7$ has weak G_2 holonomy, namely provided there exist a weak Killing spinor, satisfying the condition:

$$\mathcal{D}_I\, \eta \;=\; -\, \tfrac{3}{2}\, e\, \gamma_I\, \eta \tag{2.33}$$

we obtain a pair of G_2 forms :

$$\begin{aligned} \Phi &\equiv \phi_{IJK}\, e^I \wedge e^J \wedge e^K \;=\; \tfrac{1}{6} \eta^T\, \gamma_{IJK} \eta\, e^I \wedge e^J \wedge e^K \\ \Phi^\star &= \phi^\star_{IJKL}\, e^I \wedge e^J \wedge e^K \wedge e^L \;=\; \tfrac{1}{24} \eta^T\, \gamma_{IJKL} \eta\, e^I \wedge e^J \wedge e^K \wedge e^L \end{aligned} \tag{2.34}$$

satisfying the condition

$$d\Phi = 12\, e\, \Phi^\star \tag{2.35}$$

In this case a unique consistent Englert solution of M-theory with internal fluxes is obtained by setting

$$R^{rs}{}_{tu} \;=\; -30\, e^2\, \delta^{rs}_{tu} \tag{2.36}$$

$$\mathcal{R}^{IK}_{JK} \;=\; 27\, e^2\, \delta^I_J \tag{2.37}$$

$$F_{rstu} \;=\; e\, \epsilon_{rstu} \tag{2.38}$$

$$F_{IJKL} \;=\; 12\, e\, \phi^\star_{IJKL} \tag{2.39}$$

It must also be noted that equation (2.37) is not an independent condition but, according to the previous discussion is a consequence of eq. (2.35).

As we already mentioned in the introduction there exist several compact manifolds of weak G_2 holonomy. In particular all the coset manifolds $\mathcal{S}/\mathcal{R}$ of weak G_2 holonomy were classified and studied in the Kaluza Klein supergravity age [57, 58, 59, 60, 61] and they were extensively reconsidered in the context of the AdS/CFT correspondence [66, 67, 68, 69, 70]. No one so far formulated and answered the question whether there exist 7-parameter Lie group manifolds $\mathcal{G}$ of weak G_2 holonomy. In the next sections we address such a question and we obtain a negative answer. No such Lie group manifold exists.

3 Group theoretical reformulation of weak G_2 holonomy at the spin connection level

Let us address in this section the question whether there are twisted torii of weak G_2 holonomy, namely non semisimple group manifolds with such a property. To this effect a very useful result was obtained in paper [50] where it was shown that equation (2.35) is fully equivalent to a condition imposed directly on the spin connection of the internal manifold, namely:

$$\phi^{IJK}\,\omega^{JK} \;=q\,e^I \quad ; \quad \text{with } q=-6\,e \tag{3.1}$$

This reformulation is very useful because can be immediately translated into a group theoretical language. In order to satisfy eq. (3.1) we can just set:

$$\omega^{IJ}=\omega^{IJ}_{(14)}\;+\;6\,q\,\phi^{IJK}\,e^K \tag{3.2}$$

where $\omega^{IJ}_{(14)}$ is a one–form valued in the **14**–dimensional representation of G_2, namely in the adjoint. Being a one-form it can be expanded along the vielbein and we can write:

$$\omega^{IJ}_{(14)}=\overline{\omega}^{IJ}_{K}\,e^K \tag{3.3}$$

Group theoretically the tensor $\overline{\omega}^{IJ}_{K}$ is in the product of the **14** with the **7** and we have the decomposition:

$$\mathbf{14}\times\mathbf{7}=\mathbf{64}\oplus\mathbf{27}\oplus\mathbf{7} \tag{3.4}$$

Hence, a priori, the spin connection of a weak G_2 holonomy manifold can be parametrized with the above irreducible G_2 tensors. Let us however suppose that our manifold is a group manifold, characterized by the Maurer Cartan eq.s (1.4) satisfied by the seven vielbein where ${\tau^I}_{JK}$ are the structure constants of a seven dimensional Lie algebra spanned by generators $\{T_J\}$ which are dual to the 1-forms e^I:

$$[T_I\,,\,T_J]\;=\;{\tau^K}_{IJ}\,T_K \quad \Leftrightarrow \quad e^I\,(T_J)=\delta^I_J \tag{3.5}$$

Let us furthermore suppose that the above Lie algebra is *volume preserving*, namely satisfies the additional condition:

$${\tau^I}_{IK}=0 \tag{3.6}$$

which was assumed both in the series of papers [25, 28, 29, 31], and in [48]. In this case the representation **7** is suppressed in the decomposition (3.4). Indeed, the spin connection, defined by:

$$\partial\,e^I\;+\;\omega^{IJ}\,\wedge\,e^J\;=\;0 \tag{3.7}$$

turns out to be related to the structure constants by the simple formula:

$${\tau^I}_{AB}=\omega^{IB}_{A}-\omega^{IA}_{B} \tag{3.8}$$

and equation (3.6) simply states that the representation **7** is not present.

If instead we do not make this assumption we still have to add a seven dimensional representation. In any case by means of this argument the problem is reduced to its

algebraic core. In appendix A we recall how the **64** and **27** representations are constructed in terms of tensors satisfying certain G_2 invariant constraints. These constraints can also be algebraically solved (we did it with a computer programme in MATHEMATICA) and the corresponding tensors parametrized by 64, respectively 27 independent parameters, denoted ξ^i ($i = 1, \ldots, 64$) and α^i ($i = 1, \ldots, 64$) are named $H^{(64)}_{\substack{A \\ B}\ \ I}(\xi)$ and $U^{(27)}_{ABI}(\alpha)$. In terms of these latter we can write an ansatz for the spin connection:

$$\omega_I^{AB} = H^{(64)}_{\substack{A \\ B}\ \ I}(\xi) + U^{(27)}_{ABI}(\alpha) + 6\,q\,\phi^{ABI} \tag{3.9}$$

which, through the simple formula (3.8), leads to an ansatz for the corresponding candidate structure constants $\tau^I{}_{AB}(\xi, \alpha, q)$ depending on the 91 parameters (ξ, α) plus the parameter q. As long as $q \neq 0$ we are guaranteed to have weak G_2 holonomy. The problem is that, in order to define a true Lie algebra, the $\tau^I{}_{AB}(\xi, \alpha, q)$ should satisfy Jacobi identities:

$$\tau^I{}_{[AB}(\xi, \alpha, q)\,\tau^J{}_{C]I}(\xi, \alpha, q) = 0 \tag{3.10}$$

So we should solve the quadratic equations (3.10) and find out how many of the 91 parameters remain free. These, modulo G_2 rotations, will span the space of 7–dimensional Lie algebras of weak G_2 holonomy. At first sight 91 seems a quite comfortable number and one might expect to find not just one but several solutions. Yet Lie algebras are tough cookies and the real case is just the opposite. No solutions do exist. A direct proof by solving the quadratic equations (3.10) is conceivable but certainly requires a lot of computer time. We can however reach the same result by focusing on the Ricci tensor defined by the spin connection (3.8). The Ricci form, if the algebra had weak G_2 would be proportional to a Kronecker delta and so, in particular positive definite. In the next section, by exploiting structural Lie algebra theorems we will be able to prove that this cannot happen.

4 The Ricci tensor of metric Lie algebras and the main no-go theorem

The main object of study in the present section is the Ricci tensor of a group manifold $\mathcal{G}$, whose Lie algebra $\mathbb{G}_7$ is characterized by the structure constants $\tau^I{}_{AB}$, alternatively defined by the commutation relations (3.5) or by the Maurer Cartan equations (1.4). The metric of the manifold is given by:

$$ds^2_{\mathcal{G}} = \eta_{IJ}\, e^I \otimes e^J \tag{4.1}$$

where η_{IJ} is the flat metric. In agreement with the notations of [58], which we have adopted throughout our entire discussion, the spin connection 1-form is defined by the vanishing torsion equation:

$$de^I - \omega^{IJ} \wedge e^K\, \eta_{JK} = 0 \tag{4.2}$$

and the Riemann 2-form is normalized as follows:

$$\mathcal{R}^{AB} \equiv d\omega^{AB} - \omega^{AI} \wedge \omega^{JB} \eta_{IJ} = \mathcal{R}^{AB}{}_{PQ} e^P \wedge e^Q \tag{4.3}$$

so that the Ricci tensor is finally defined by:

$$\mathrm{Ric}[\tau]^I{}_J \equiv \mathcal{R}^{IM}{}_{JM} \tag{4.4}$$

By explicit calculation, from the above formulae (4.2-4.4) one obtains

$$\begin{aligned} \mathrm{Ric}[\tau]^I{}_J \;=\; & \frac{1}{4}\Big(\eta^{IN}\,\tau^K{}_{NS}\,\tau^S{}_{JK} + \eta^{IQ}\,\eta^{SM}\,\eta_{PN}\,\tau^P{}_{QS}\,\tau^N{}_{JM} - \frac{1}{2}\eta^{PQ}\,\eta^{SM}\,\eta_{JN}\,\tau^I{}_{PS}\,\tau^N{}_{QM} \\ & + \eta^{NK}\,\tau^I{}_{NJ}\,\tau^S{}_{KS} - \eta^{IN}\,\tau^K{}_{NJ}\,\tau^S{}_{KS} + \eta^{PQ}\,\eta^{IM}\,\eta_{JN}\,\tau^S{}_{PS}\,\tau^N{}_{QM}\Big) \end{aligned} \tag{4.5}$$

where we have included also the trace terms of type $\tau^S{}_{PS}$, which means that we have relaxed the area preserving condition (3.6). In our case of interest, namely in compactifications of M-theory the flat metric η^{IJ} is chosen to be the negative Euclidean metric:

$$\eta^{IJ} = -\,\delta^{IJ} \tag{4.6}$$

This corresponds to the conventions of [58] and follows from the fact that the 11–dimensional signature was chosen mostly minus. When eq.(4.6) is adopted eq. (4.5) can be rewritten in a very convenient matrix form:

$$\begin{aligned} \mathrm{Ric}[\tau]^I{}_J \;=\; & -\tfrac{1}{4}\,\mathrm{Tr}\left[\tau_I\,\tau_J\right] - \tfrac{1}{4}\mathrm{Tr}\left[\tau_I\,\tau_J^T\right] + \tfrac{1}{8}\sum_{K=1}^{7}\left(\tau_K\,\tau_K^T\right)^{IJ} \\ & - \tfrac{1}{4}\sum_{K=1}^{7}\tau^I{}_{KJ}\,\mathrm{Tr}\left[\tau_K\right] + \tfrac{1}{4}\sum_{K=1}^{7}\tau^K_{\ IJ}\,\mathrm{Tr}\left[\tau_K\right] - \tfrac{1}{4}\sum_{K=1}^{7}\tau^J{}_{KI}\,\mathrm{Tr}\left[\tau_K\right] \end{aligned} \tag{4.7}$$

where $(\tau_I)^A_B \equiv \tau^A{}_{IB}$ is the matrix representing the generator T_I in the adjoint representation and $\sum_{K=1}^{7}\left(\tau_K\,\tau_K^T\right)^{IJ}$ denotes the matrix $\sum_{K,L=1}^{7}\tau^I{}_{KL}\,\tau^J{}_{KL}$.

As we have extensively discussed in previous sections, the problem of establishing whether or not, there exist *bona fide* compactifications of M-theory on twisted tori with non trivial fluxes has been reduced, in the absence of sources, to the question whether or not, there exist 7–dimensional Lie algebras of weak G_2 holonomy. Relying on the general representation of the Ricci tensor provided by eq.(4.7) we can now state our main negative result as the following:

No go Theorem 4.1 *There exists no real 7-dimensional metric Lie algebra $\mathbb{G}_7$ of weak G_2 holonomy*

Proof 4.1.1 We prove this theorem in a series of steps by exhaustion of all possible cases. First let us explain the general strategy of the proof. Given a real Lie algebra $\mathbb{G}_7$ identified, in any given basis of generators $\{T_I\}$, by its structure constants τ^I_{JK}, normalized as in eq.(3.5), a metric $\mathbf{g}$ on $\mathbb{G}_7$ is a bilinear, symmetric, non degenerate, negative definite 2-form:

$$<\,,\,>_{\mathbf{g}} \quad : \quad \mathbb{G} \otimes \mathbb{G} \to \mathbb{R} \tag{4.8}$$

As usual $< , >_{\mathbf{g}}$ is determined by giving its values in a basis, namely on the generators $\{T_I\}$:

$$< T_I \, , \, T_J >_{\mathbf{g}} = \, \mathbf{g}_{IJ} \tag{4.9}$$

By hypothesis the matrix $\mathbf{g}_{IJ}$ is symmetric ($\mathbf{g}_{IJ} = \mathbf{g}_{JI}$), non-degenerate ($\det \mathbf{g} \neq 0$) and negative definite, namely all of its eigenvalues are strictly negative ($\lambda_I < 0$). The holonomy, in particular weak G_2 holonomy, is a property of the metric, so that on the same algebra $\mathbb{G}_7$ there can be a metric $\mathbf{g}_{IJ}$ with a certain holonomy and another one $\widetilde{\mathbf{g}}_{IJ}$ with a different holonomy. The crucial observation, however, is that, once the signature of $\mathbf{g}_{IJ}$ has been fixed (in our case the euclidean negative one), the choice of the metric is equivalent to a change of basis in the algebra. Indeed under the latter $T_I \rightarrow \mathcal{A}_I{}^J \, T_J$, where $\det\mathcal{A} \neq 0$ is a non degenerate real matrix, the metric changes as follows:

$$\mathbf{g} \, \rightarrow \, \widetilde{\mathbf{g}} \, = \, \mathcal{A}^T \, \mathbf{g} \, \mathcal{A} \tag{4.10}$$

and by means of a suitable real $\mathcal{A}$ we can always obtain $\widetilde{\mathbf{g}}_{IJ} = -\delta_{IJ}$. As it is evident from its definition (4.7) also the Ricci form changes in the same way as the metric:

$$\mathbf{Ric}[\tau] \, \rightarrow \, \mathbf{Ric}[\widetilde{\tau}] \, = \, \mathcal{A}^T \, \mathbf{Ric}[\tau] \, \mathcal{A} \tag{4.11}$$

So that, instead of considering the space of metrics on each given 7–dimensional Lie algebra $\mathbb{G}_7$, it is sufficient, fixing the metric to be the standard one, (*i.e.* $\mathbf{g}_{IJ} = -\delta_{IJ}$), to consider all possible Lie algebra bases, parametrized by the elements $\mathcal{A} \in \mathrm{GL}(7, \mathbb{R})$ of the general linear group in 7–dimensions. As we have recalled in previous sections, if the spin connection of a 7–manifold $\mathcal{M}_7$ has weak G_2–holonomy, namely if ω^{IJ} satisfies the defining condition (3.1), then the intrinsic Ricci tensor $\mathbf{Ric}^I{}_J \equiv \mathcal{R}^{IM}{}_{JM}$ is automatically proportional to $\delta^I{}_J$ with a positive coefficient, namely it is a positive definite symmetric 2-form. This observation provides a very severe necessary (although not yet sufficient condition) for weak G_2 holonomy: the Ricci form of the candidate $\mathbb{G}_7$ Lie algebra, equipped with a negative definite metric $\mathbf{g} < 0$ should instead be positive definite $\mathbf{Ric} > 0$. The value of this criterion is that it depends only on the choice of the algebra and not on the choice of the basis, or equivalently of the metric. So if we are able to prove that for all 7-dimensional Lie algebras the Ricci form is never positive definite, then we have proved our theorem.

As we anticipated, we proceed by exhaustion of all the possible cases, relying on the fundamental structural theorem by Levi which states that

Levi Theorem 4.1 *Any Lie algebra* $\mathbb{G}$ *of dimension* $\dim \mathbb{G} = n$ *is the semidirect product of a semisimple Lie algebra* $\mathbb{L}(\mathbb{G})$ *of dimension* $\dim \mathbb{L}(\mathbb{G}) = m$*, called the Levi subalgebra* $\mathbb{L}(\mathbb{G}) \subset \mathbb{G}$ *with a solvable ideal* $\mathrm{Rad}(\mathbb{G}) \subset \mathbb{G}$ *of dimension* $\dim \mathrm{Rad}(\mathbb{G}) = q$ *so that* $n = m + q$*. The solvable ideal* $\mathrm{Rad}(\mathbb{G})$ *is named the radical of* $\mathbb{G}$

For the proof of Levi's theorem we refer the reader to standard textbooks as [71] or [72]. Applying it to our case we can just go by dimension of the radical, namely by values of q.

($q = 0$) This corresponds to the case when $\mathrm{Rad}(\mathbb{G}_7) = 0$, namely when $\mathbb{G}_7$ is semisimple. Yet there are no semisimple Lie algebras of dimension 7, so this case is already ruled out.

($q = 7$) This is the extreme opposite case when the Levi subalgebra vanishes $\mathbb{L}(\mathbb{G}_7) = 0$, namely $\mathbb{G}_7$ is completely solvable. By definition this means that the first derivative of the algebra $\mathcal{D}\mathbb{G}_7 \equiv [\mathbb{G}_7 \, , \, \mathbb{G}_7]$ is necessarily a proper subspace *i.e.* $\dim \mathcal{D}\mathbb{G}_7 < 7$. Henceforth we can write the following orthogonal decomposition of $\mathbb{G}_7$:

$$\mathbb{G}_7 = \mathbb{K}_0 \oplus \mathcal{D}\mathbb{G}_7 \tag{4.12}$$

where $\dim\mathbb{K}_0 \geq 1$. Let us accordingly subdivide the index range as it follows $I = \{i, \alpha\}$ where $i =, 1 \ldots, \dim \mathbb{K}_0$ spans the 0-grading subspace while $\alpha = \dim \mathbb{K}_0 + 1\, , \ldots, 7$ spans the ideal $\mathcal{D}\mathbb{G}_7$. In these notations we have that the structure constants $\tau^i{}_{IJ} = 0$ vanish for all values of the lower indices IJ. Considering next eq.(4.7) we can calculate the entries ij of the Ricci form. We immediately find:

$$\mathbf{Ric}^i{}_j = -\frac{1}{2} \,\mathrm{Tr}\, \left(\tau^{(S)}_i \, \tau^{(S)}_j\right) \tag{4.13}$$

where by $\tau^{(S)}_i$ we have denoted the symmetric part of the adjoint matrix $\tau^P{}_{iQ}$ representing the generator $T_i \in \mathbb{K}_0$. That eq.(4.13) is correct is easily understood by means of the following argument. The first two terms in the r.h.s of eq. (4.7) combine into the r.h.s of (4.13), all the other terms vanish since the index i or j appears as choice of the upper index of $\tau^I{}_{JK}$. Hence if calculate the Ricci norm of a vector lying in the grading 0-subspace $X = c^i \, T_i$ we find that it is strictly non positive:

$$\forall X \in \mathbb{K}_0 \quad : \quad \mathbf{Ric}(X, X) = -\frac{1}{4} \,\mathrm{Tr}\, \left(\tau^{(S)}_X\right)^2 < 0 \tag{4.14}$$

This result suffices to prove that **Ric** cannot be a positive definite 2–form and to rule out also this case.

($q = 2$) Is ruled out because there are no semisimple Lie algebras of dimension 5 and hence the Levi subalgebra cannot exist for $m = 7 - 2$

($q = 3$) Is ruled out because there are no semisimple Lie algebras of dimension 4 and hence the Levi subalgebra cannot exist for $m = 7 - 3$

($q = 5$) Is ruled out because there are no semisimple Lie algebras of dimension 2 and hence the Levi subalgebra cannot exist for $m = 7 - 5$

($q = 4$) In this case the Levi subalgebra $\mathbb{L}(\mathbb{G}_7)$ has dimension $m = 3$ and there are just two possible real simple algebras of that dimensions namely, either $\mathbb{L}(\mathbb{G}_7) = \mathrm{SO}(3)$ or $\mathbb{L}(\mathbb{G}_7) = \mathrm{SL}(2, \mathbb{R})$. In appendix B we classify by exhaustion all the seven dimensional algebras with Levi subalgebra either SO(3) (in subsection B.1 and relative subsubsections) or $\mathrm{SL}(2, \mathbb{R}) \sim \mathrm{SO}(1, 2)$ (in subsection B.2 and relative subsubsections). For each of these algebras we calculate the Ricci form and we show that in every case it has indefinite signature (both positive and non positive eigenvalues) so that all cases are excluded. A general observation is that compact generators lead to non negative eigenvalues of the Ricci form while non compact semisimple or nilpotent generators lead to non positive ones.

$(q=1)$ In this case the Levi subalgebra is either of the following three cases:

$$\mathbb{L}(\mathbb{G}_7) = \begin{cases} \mathrm{SO}(3) \oplus \mathrm{SO}(3) \sim \mathrm{SO}(4) \\ \mathrm{SO}(3) \oplus \mathrm{SO}(1,2) \\ \mathrm{SO}(1,2) \oplus \mathrm{SO}(1,2) \sim \mathrm{SO}(2,2) \end{cases} \tag{4.15}$$

and the algebra is just the direct sum of its Levi subalgebra with its 1-dimensional radical since there are no 1-dimensional representations of $\mathbb{L}(\mathbb{G}_7)$ except the singlet. Hence the Ricci form is block-diagonal $3+3+1$ and the last generator, the singlet contributes a zero eigenvalue. Hence also this last case is excluded.

This concludes the proof of our non go theorem. ■

Before ending the present section we present here a second formal proof of the main Theorem. The Ricci tensor $\mathbf{Ric}^I{}_J$ in our case is a symmetric 7×7 matrix. If it were positive definite, then, for any non–vanishing seven dimensional vector $\mathbf{V}$ we would have:

$$\mathbf{V}^T\,\mathbf{Ric}\,\mathbf{V} \quad > \quad 0 \tag{4.16}$$

Below we show that there always exists a $\mathbf{V}\neq 0$ so that $\mathbf{V}^T\,\mathbf{Ric}\,\mathbf{V} \leq 0$. This implies in turn that $\mathbf{Ric}^I{}_J$ cannot be positive definite and proves Theorem 4.1. Let us start considering the case in which the structure constants τ are traceless $\tau^I{}_{JI}=0$, which is relevant to the compactifications on twisted–tori studied in the literature.

$\tau^I{}_{JI}=0$ **case:** Let us decompose $\mathrm{Rad}(\mathbb{G}_7)$ as follows:

$$\mathrm{Rad}(\mathbb{G}_7) \quad = \quad \mathbb{K}_0 + \mathcal{D}(\mathbb{G}_7) \tag{4.17}$$

and label by $u,v=1,\dots,\dim(\mathbb{L}(\mathbb{G}_7))$ the generators of the Levi subalgebra, by $i,j\dots$ the generators of $\mathbb{K}_0$ and by $p,q\dots$ the generators of $\mathrm{Rad}(\mathbb{G}_7)$. Then the only non–vanishing entries of $\tau^i{}_{JK}$ are $\tau^i{}_{up}$ and a basis of $\mathbb{L}(\mathbb{G}_7)$ can be chosen for which the following properties hold:

$$\tau^p{}_{uq}\tau^p{}_{vq} \quad = \quad \tau^q{}_{up}\tau^q{}_{vp} \tag{4.18}$$

Let us compute the entries $\mathbf{Ric}^i{}_j$. The (positive) contribution from the third term of (4.7), namely $(1/4)\sum_{u,p=1}^7 \tau^i{}_{up}\,\tau^j{}_{up}$ is cancelled by an opposite contribution $-\frac{1}{4}\sum_{u,p=1}^7 \tau^p{}_{ui}\,\tau^p{}_{uj}$ coming from the first two terms, in virtue of eq. (4.18). We are left with the following matrix:

$$\mathbf{Ric}^i{}_j \quad = \quad -\frac{1}{4}\sum_{p,q=1}^{7} \tau^p{}_{iq}\,(\tau^p{}_{jq}+\tau^q{}_{jp}) \leq 0 \tag{4.19}$$

Therefore, if we choose the following vector $\mathbf{V}$:

$$\mathbf{V} \quad = \quad (0,\dots,0,V_i,0,\dots,0) \tag{4.20}$$

we have $\mathbf{V}^T\,\mathbf{Ric}\,\mathbf{V} \leq 0$ and the Theorem is proven.

$\tau^I{}_{JI} \neq 0$ **case:** If $\mathbb{L}(\mathbb{G}_7)$ is compact, the τ_u matrices are antisymmetric and $\mathrm{Tr}(\tau_u) = 0$. Moreover, since the commutator of two elements of $\mathbb{K}$ expands in nilpotent generators belonging to $\mathcal{D}(\mathbb{G}_7)$, whose trace therefore vanishes, we also have $\tau^K{}_{ij}\,\mathrm{Tr}(\tau_K) = 0$. In this case the terms in $\mathbf{Ric}^i{}_j$ depending on $\mathrm{Tr}(\tau_I)$ do not contribute and thus we can choose the same seven–vector $\mathbf{V}$ as in (4.20) to prove that $\mathbf{Ric}$ is not positive definite.

In the case in which $\mathbb{L}(\mathbb{G}_7)$ is a non–compact semisimple algebra, let us label by u_a and u_s the compact and non–compact generators respectively. We can have $\mathrm{Tr}(\tau_{u_s}) \neq 0$ and the previous proof would not hold. Let us consider $\mathbf{Ric}^{u_s}{}_{v_s}$ instead. The only non vanishing entries of τ having one index u_s are:

$$\tau^{u_a}{}_{u_s v_s} = \tau^{v_s}{}_{u_s u_a}\ ;\ \tau^p{}_{u_s q} = \tau^q{}_{u_s p} \tag{4.21}$$

The contribution from the last three terms of (4.7) vanishes since it involves only $\mathrm{Tr}(\tau_{u_a})$. The only positive term (the third) in (4.7) reads $\frac{1}{4}\sum_{u_a w_s=1}^{7} \tau^{u_s}{}_{u_a w_s}\,\tau^{v_s}{}_{u_a w_s}$ and it is cancelled by an opposite contribution from the first two terms. The remaining matrix is non–positive: $\mathbf{Ric}^{u_s}{}_{v_s} \leq 0$ since it has contributions only from the first two terms and thus it suffices to choose the seven–vector $\mathbf{V}$ with non–vanishing entries only along the non–compact generators of the Levi subalgebra to have $\mathbf{V}^T\,\mathbf{Ric}\,\mathbf{V} \leq 0$.

5 Conclusions

In this paper we have proved that no Lie group 7-manifolds of weak G_2 holonomy do exist. We have also shown that the notion of weak G_2 holonomy is identical with the notion of Englert 7-manifolds introduced and used in the literature on Kaluza Klein supergravity two decades ago. Our analysis rules out the possibility of introducing on a bosonic background a non trivial 4–form flux, and thus, in the light of the results obtained in [48], it follows that on these solutions the minimal part of the FDA coincides with the algebra $\mathbb{G}_7$ gauged by the Kaluza–Klein vectors. The triviality of the 4–form flux on bosonic backgrounds was also found in [31, 36]. As a byproduct of our analysis we rule out also (unwarped, supersymmetric and non–supersymmetric) anti–de Sitter four dimensional vacua (Freund–Rubin) from M–theory compactifications on the so called "twisted" tori. Yet, as we have spelled out in the introduction, this no go theorem applies only to the case of the field equations of M–theory in the bulk, with no source terms (no M2 or M5 branes) and no warp factor. It is then obvious that the hunting ground is singled out by our analysis. One should

a Repeat the cohomological analysis of papers [25, 28, 29, 31] and [48] in the presence of a warp factor.

b Analyze the extent to which a warp factor can cope for the absence of a Lie algebra of weak G_2 holonomy

c Analyze the extent to which the $M2$ and $M5$ brane contributions to the stress energy tensor can help in introducing consistent internal fluxes.

These are the guidelines for any further investigation.

It is also worth mentioning that a similar analysis for twisted tori compactifications of Type IIB theory is still missing and should certainly be considered.

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A G_2 formalism: notations and normalizations

The group G_2 is defined as the stability subgroup of an SO(7) spinor so that we have the branching rule:

$$\mathbf{8} \xrightarrow{G_2} \mathbf{7} \oplus \mathbf{1} \tag{A.1}$$

Correspondingly the fundamental vector representation of SO(7):

$$\mathbf{7} \xrightarrow{G_2} \mathbf{7} \tag{A.2}$$

remains instead irreducible. For this reason we begin by constructing a set of gamma matrices satisfying the SO(7) Clifford algebra in the form:

$$\{\gamma_I \, , \, \gamma_J\} = -2\,\delta_{IJ} \tag{A.3}$$

The minus sign in eq.(A.3) is due to our choice of a mostly minus metric in the conventions for M–theory, so that the internal 7–dimensions have a negative Euclidean metric. This has special advantages. Indeed with the choice (A.3) the gamma matrices are all real and antisymmetric. Indeed an explicit realization of this Clifford algebra, given by real antisymmetric matrices:

$$\gamma_I^\star = \gamma_I \quad ; \quad \gamma_I^T = -\gamma_I \tag{A.4}$$

is given below:

$$\begin{aligned}
\gamma_1 &= \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{pmatrix} \; ; \;
\gamma_2 = \begin{pmatrix} 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0 \end{pmatrix}\\
\gamma_3 &= \begin{pmatrix} 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix} \; ; \;
\gamma_4 = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix}\\
\gamma_5 &= \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0 \end{pmatrix} \; ; \;
\gamma_6 = \begin{pmatrix} 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{pmatrix}\\
\gamma_7 &= \begin{pmatrix} 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&-1&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0 \end{pmatrix}
\end{aligned} \tag{A.5}$$

Let e^I be the vielbein of the considered 7-dimensional space. As it is well known we can construct a G_2-invariant three-form (defining the G_2–structure of the corresponding 7-manifold) by writing:

$$\Phi = \frac{1}{6}\,\eta^T\,\gamma_{IJK}\,\eta\,e^I\wedge e^J\wedge e^K \equiv \phi_{IJK}\,\eta\,e^I\wedge e^J\wedge e^K \tag{A.6}$$

where η is the G_2 invariant 8-component commuting spinor which, by choice of normalization, we take to be the following one:

$$\eta = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix} \tag{A.7}$$

With this choice we can calculate the explicit expression of the G_2 three-form and we obtain:

$$\Phi = e^1\wedge e^2\wedge e^7 + e^1\wedge e^3\wedge e^5 - e^1\wedge e^4\wedge e^6 - e^2\wedge e^3\wedge e^6 - e^2\wedge e^4\wedge e^5 + e^3\wedge e^4\wedge e^7 + e^5\wedge e^6\wedge e^7 \tag{A.8}$$

The reader can observe that we have calibrated the form of our gamma matrices (A.3) in such a way that the expression for Φ coincides with that usually adopted in the mathematical literature on G-structures (see for instance [73]). The dual G_2 invariant four–form is defined by:

$$\Phi_\star \equiv \frac{1}{24}\,\eta^T\,\gamma_{IJKL}\,\eta\,e^I\wedge e^J\wedge e^K\wedge e^L \;\equiv\; \phi^\star_{IJKL}\,e^I\wedge e^J\wedge e^K\wedge e^L \tag{A.9}$$

and its explicit expression is the following:

$$\begin{aligned}\Phi_\star \;=\; & -e^1\wedge e^2\wedge e^3\wedge e^4 - e^1\wedge e^2\wedge e^5\wedge e^6 - e^1\wedge e^3\wedge e^6\wedge e^7 - e^1\wedge e^4\wedge e^5\wedge e^7 \\ & -e^2\wedge e^3\wedge e^5\wedge e^7 + e^2\wedge e^4\wedge e^6\wedge e^7 - e^3\wedge e^4\wedge e^5\wedge e^6 \end{aligned} \tag{A.10}$$

The components of these invariant G_2 forms are dual to each other since they satisfy the relation:

$$-\frac{1}{6}\,\epsilon_{IJKABCD}\,\phi^\star_{ABCD} \;=\; \phi_{IJK} \tag{A.11}$$

Next we introduce the projection operators onto the **14** and the **7** dimensional representations of G_2. We recall that the adjoint of SO(7) splits in the following way with respect to the G_2 subalgebra:

$$\mathbf{21} \xrightarrow{G_2} \underbrace{\mathbf{14}}_{adjoint} \oplus \underbrace{\mathbf{7}}_{fundamental} \tag{A.12}$$

The adjoint of SO(7) is given by an antisymmetric tensor $M^{IJ} = -M^{JI}$ whose decomposition is as follows:

$$M^{IJ} = M_{14}^{IJ} + M_7^{IJ} \tag{A.13}$$

The **14** part is defined by the G_2 invariant condition:

$$\phi^{IJK} M_{14}^{JK} = 0 \tag{A.14}$$

Correspondingly the projector onto the **14** part of any antisymmetric tensor is given by:

$$P_{(14)}^{IJ}{}_{KL} = \frac{2}{3}\delta_{KL}^{IJ} + 4\phi_\star^{IJKL} \tag{A.15}$$

while the projector onto the **7** part is given by:

$$P_{(7)}^{IJ}{}_{KL} = \frac{1}{3}\delta_{KL}^{IJ} - 4\phi_\star^{IJKL} \tag{A.16}$$

Applied onto the two representations the dual form $\phi_\star^{IJKL}$ is a diagonal operator with two different eigenvalues:

$$\begin{aligned} \phi_\star^{IJKL} M_{14}^{KL} &= \frac{1}{12} M_{14}^{IJ} \\ \phi_\star^{IJKL} M_7^{KL} &= -\frac{1}{6} M_7^{IJ} \end{aligned} \tag{A.17}$$

The expression of these projection operators and the calculation of the eigenvalues (A.17) was presented long time ago in [51]. More recently appeared often in the literature on flux compactifications and specifically in [50].

A.1 A Fierz identity and the G_2 decomposition of the 35 representation of SO(7)

An important relation is provided by the following identity (see for instance [50]):

$$\begin{aligned} \phi_\star^{ABCX} \phi^\star_{IJKX} = & -\frac{3}{8}\phi_\star^{AB[IJ}\delta^{K]C} \\ & -\frac{1}{16}\phi^{ABC}\phi_{IJK} + \frac{1}{96}\delta_{IJK}^{ABC} \end{aligned} \tag{A.18}$$

which is used to perform the decomposition of the **35** representation of SO(7) into its irreducible G_2 parts.

The **35** representation of SO(7) is given by an antisymmetric tensor of rank three. We have:

$$U^{IJK} \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \tag{A.19}$$

Alternatively, from the SO(7) point of view, an antisymmetric tensor can be seen as a traceless, symmetric bispinor. Indeed, we have that the 8×8 three-index gamma matrices:

$$(\gamma_{IJK})_{\alpha\beta} \tag{A.20}$$

are symmetric and traceless.

The decomposition of the **35** with respect to $G_2 \subset SO(7)$ is as follows:

$$\mathbf{35} \xrightarrow{G_2} \mathbf{27} \oplus \mathbf{7} \oplus \mathbf{1} \tag{A.21}$$

It is interesting and necessary for our goals to construct the projectors onto these representations.

The singlet representation of G_2 is given by the 3-tensors U^{IJK} proportional to the G_2-invariant 3–form, namely:

$$U^{IJK}_{(1)} \propto \phi^{IJK} \tag{A.22}$$

It is therefore easy to write the projection operator onto the singlet:

$$P^{IJK}_{(1)\ \ ABC} = \frac{6}{7}\phi_{ABC}\phi^{IJK} \tag{A.23}$$

which has the property :

$$P^{IJK}_{(1)\ \ ABC}\,\phi_{IJK} = \phi_{ABC} \tag{A.24}$$

Next we are interested in the projection operator onto the **7**–representation. We observe that if we define the two operators:

$$K_{\mp} = \begin{cases} \delta^{IJK}_{ABC} - 24\,\phi_\star^{IJKX}\,\phi^\star_{ABCX} \\ 24\,\phi_\star^{IJKX}\,\phi^\star_{ABCX} \end{cases} \tag{A.25}$$

both of them are projection operators. Indeed:

$$(K_{\mp})^2 = K_{\mp} \tag{A.26}$$

Then we observe that defining:

$$P^{IJK}_{(7)\ \ ABC} \equiv 24\,\phi_\star^{IJKX}\,\phi^\star_{ABCX} = K_+ \tag{A.27}$$

we have:

$$P^{IJK}_{(7)\ \ ABC}\,P^{XYZ}_{(1)\ \ IJK} = 0 \tag{A.28}$$

and we deduce that indeed the operator in eq.(A.27) is the projection operator onto the **7**–dimensional representation.

We consider next the projection operator onto the **27**–representation. We set:

$$P^{IJK}_{(27)\ \ ABC} = 36\,\phi_\star^{[IJ}{}_{[AB}\,\delta^{K]}_{C]} + 72\,\phi_\star^{IJKX}\,\phi^\star_{ABCX} + \frac{36}{7}\phi_{ABC}\,\phi^{IJK} \tag{A.29}$$

Then we verify that:

$$P^2_{(27)} = P_{(27)} \quad ; \quad P_{(27)}\,P_{(1)} = 0 \quad ; \quad P_{(27)}\,P_{(1)} = 0 \tag{A.30}$$

Moreover thanks to the Fierz identity (A.18) we have:

$$P_{(27)} + P_{(7)} + P_{(1)} = \mathbf{1} \tag{A.31}$$

In this way we have verified that the three projectors perform a splitting of the **35** representation of SO(7) into three orthogonal subspaces corresponding to the three irreducible representations of G_2.

A.2 Spinorial description of the 27 representation

As we have already recalled the group G_2 is defined as the stability subgroup of an SO(7) spinor (see eq. (A.1)). The fundamental vector representation of SO(7) remains instead irreducible (see eq.(A.2)). The same happens for the symmetric tensor representation of SO(7). A symmetric traceless two-tensor of SO(7):

$$M_{IJ} \sim \overset{\circ}{\square\square} \quad ; \quad M_{IJ} = M_{JI} \quad ; \quad M_{JJ} = 0 \tag{A.32}$$

has 27 independent components and constitutes an irreducible representation. Under G_2 reduction this representation remains irreducible:

$$\mathbf{27} \xrightarrow{G_2} \mathbf{27} \tag{A.33}$$

so that the **27** of G_2 has an alternative simpler description in terms of a symmetric traceless tensor. What is the relation between this and the previous description? It is as usual provided by gamma matrices. Let us see.

The result of our previous discussion is that an irreducible **27** representation of G_2 is provided by an antisymmetric tensor U_{27}^{IJK} satisfying the two G_2 invariant constraints:

$$\begin{aligned} 0 &= \phi_\star^{ABCT} \, U_{(27)}^{ABC} \\ 0 &= \phi^{ABC} \, U_{(27)}^{ABC} \end{aligned} \tag{A.34}$$

which respectively eliminate the **7** and **1** irreducible representations.

Let us now consider the spinor index $\alpha = 1, \ldots, 8$ of SO(7) and split it as follows:

$$\alpha = \begin{cases} A = 1, \ldots, 7 \\ 8 \end{cases} \tag{A.35}$$

which corresponds to the branching rule (A.1). Correspondingly, the symmetric, traceless 8×8 matrices (A.20) split as follows:

$$\gamma_{\alpha\beta}^{IJK} = \begin{cases} \gamma_{AB}^{IJK} \\ \gamma_{A8}^{IJK} \\ \gamma_{88}^{IJK} \end{cases} \tag{A.36}$$

where

$$\gamma_{AA}^{IJK} = -\gamma_{88}^{IJK} \tag{A.37}$$

follows from $\gamma_{\alpha\beta}^{IJK}$ being traceless.

The decomposition (A.36) is the decomposition of the **35** representation of SO(7) into its **27**, **7** and **1** components with respect to G_2. Indeed we have that:

$$\gamma_{88}^{IJK} = 6\phi^{IJK} \tag{A.38}$$

while:

$$P_{(7)}^{IJK}{}_{ABC} \, \gamma_{T8}^{ABC} = \gamma_{T8}^{IJK} \tag{A.39}$$

Finally defining the traceless matrices:

$$W_{XY}^{IJK} = \gamma_{XY}^{IJK} - \frac{1}{7}\,\delta_{XY}\gamma_{TT}^{IJK} \tag{A.40}$$

we can verify that:

$$P_{(27)}^{IJK}{}_{ABC}\, W_{XY}^{ABC} = W_{XY}^{IJK} \tag{A.41}$$

which proves what we just stated.

A.3 The representation 64

Let us now define the representation 64 of G_2.

In the case of SO(7) we can consider an irreducible traceless tensor with gun symmetry

$$H_{\substack{A\\B}\;C} \sim \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{A.42}$$

The tensor H is antisymmetric in AB:

$$H_{\substack{A\\B}\;C} = -\,H_{\substack{B\\A}\;C} \tag{A.43}$$

fulfills the cyclic identity:

$$H_{\substack{A\\B}\;C} + H_{\substack{C\\A}\;B} + H_{\substack{B\\C}\;A} = 0 \tag{A.44}$$

and it is traceless:

$$H_{\substack{A\\B}\;B} = 0 \tag{A.45}$$

The independent components of such a tensor are **105** and they constitute an irreducible representation of SO(7). The branching rule of this representation with respect to G_2 is the following one:

$$\mathbf{105} \xrightarrow{G_2} \mathbf{64} + \mathbf{14} + \mathbf{1} \tag{A.46}$$

How do we understand it? In the following way. Given the tensor H, by means of the G_2 invariant form we can define the following matrix:

$$\mathcal{C}_{AB} = \phi_{AIJ}\, H_{\substack{I\\J}\;B} \tag{A.47}$$

Decomposing this matrix into its symmetric and antisymmetric parts:

$$\begin{aligned} \mathcal{A}_{AB} &\equiv \frac{1}{2}\left(\mathcal{C}_{AB} - \mathcal{C}_{BA}\right) \\ \mathcal{S}_{AB} &\equiv \frac{1}{2}\left(\mathcal{C}_{AB} + \mathcal{C}_{BA}\right) \end{aligned} \tag{A.48}$$

we can easily verify that:

$$\begin{aligned} P_{(14)}^{IJ} \quad {}_{AB}\,\mathcal{A}_{IJ} &= \mathcal{A}_{AB} \\ S_{AA} &= 0 \end{aligned} \tag{A.49}$$

This shows that imposing the constraint:

$$0 = \phi_{AIJ}\, H_{\ I\ \ B}^{\ \ J} \tag{A.50}$$

is just necessary and sufficient to remove the **27** and **14** irreducible representation of G_2 and leave a pure **64** representation. In other words a pure 64 tensor is a tensor H_{ABC} satisfying all the constraints (A.43), (A.44), (A.45) and (A.50).

B 7-dimensional Lie algebras $\mathbb{G}_7$ with $\dim \mathrm{Rad}(\mathbb{G}_7) = 4$

In this appendix we describe the explicit structure of those 7–dimensional Lie algebras whose Levi subalgebra has dimension 3, or, equivalently the radical has dimension 4.

B.1 Algebras $\mathbb{G}_7$ with Levi subalgebra $\mathbb{L}(\mathbb{G}_7) = \mathrm{SO}(3)$

The bosonic representations of SO(3), namely the representations $j = n \in \mathbb{Z}_+$ are real, but, due their dimensionality $2j+1$ they are all odd dimensional. Hence the only bosonic real representations of SO(3) with dim = 4 are either four singlets leading to a radical of the following form

$$\mathrm{Rad}(\mathbb{G}_7) = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \tag{B.1}$$

or one singlet plus one triplet leading to a radical of the following form:

$$\mathrm{Rad}(\mathbb{G}_7) = \mathbf{1} \oplus \mathbf{3} \tag{B.2}$$

B.1.1 The case of a triplet plus a singlet

In case (B.2) naming the seven generators as follows:

$$\{J_1, J_2, J_3, W_1, W_2, W_3, Z\} \tag{B.3}$$

where J_x are the SO(3) generators and W_x the triplet generators, while Z is the singlet, the most general form of the commutation relations is the following one:

$$\begin{aligned} [J_x\,,\,J_y] &= \epsilon_{xyz}\,J_z \\ [J_x\,,\,W_y] &= \epsilon_{xyz}\,W_z \\ [J_x\,,\,Z] &= 0 \\ [W_x\,,\,W_y] &= 0 \\ [Z\,,\,W_x] &= \gamma\,W_x \end{aligned} \tag{B.4}$$

We name this algebra **so3w3** + **1** and calculating its Ricci form we obtain:

$$\mathbf{Ric}_{so3w3+1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-3\gamma^2}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-3\gamma^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-3\gamma^2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3\gamma^2}{2} \end{pmatrix} \tag{B.5}$$

which has both positive and negative eigenvalues, leading to the conclusion that this algebra is ruled out as a candidate for weak G_2 holonomy.

B.1.2 The case of 4 singlets

In this case the algebra $\mathbb{G}_7$ is not the semidirect product of its Levi subalgebra with its radical but just the *direct* sum of the same:

$$\mathbb{G}_7 = \mathbf{SO(3)} \oplus \mathrm{Rad}_(\mathbb{G}_7) \tag{B.6}$$

Correspondingly the Ricci form is just block diagonal $3+4$. The contribution of the Levi algebra $\mathbf{SO(3)}$ is just a positive definite matrix, actually $\frac{1}{4} \times \mathbf{1}_{3\times 3}$, while, by definition, $\mathrm{Rad}(\mathbb{G}_7)$ is a completely solvable 4-dimensional Lie algebra. Then, for $\mathrm{Rad}(\mathbb{G}_7)$, it is true what in the main text we already proved for any solvable Lie algebra (see eq.s (4.12), (4.13 and related text), namely that the Ricci form has at least one non–positive eigenvalue. By means of this argument also this case is excluded, leading to an overall Ricci form non positive definite. It remains to analyze the last case.

B.1.3 The algebra so3w4

This Lie algebra is the semidirect product of the rotation algebra SO(3) with its only available real 4-dimensional irreducible representation, namely the real transcription of the $j = \frac{1}{2}$ spinor representation. The reasoning leading to this algebra is the following one. It remains the case of spinor representations $j = \frac{n}{2}$ with $n \in \mathbb{Z}_+$. These are all complex representations and they can be transcribed as real representations in twice their complex dimensions namely in $4j + 2$ dimensions. Hence the only possibility is $j = \frac{1}{2}$.

To this effect we consider the 4×4 't Hooft matrices which constitute two triplets of either self-dual or antiself dual SO(3) generators, that commute with each other:

$$\begin{aligned} \mathbf{J}^{\pm|x}_{ij} &= -\mathbf{J}^{\pm|x}_{ji} \quad : \quad x = 1,2,3 \quad ; \quad i,j = 1,2,3,4 \\ \mathbf{J}^{\pm|x}_{ij} &= \pm\tfrac{1}{2}\epsilon_{ijkl}\,\mathbf{J}^{\pm|x}_{kl} \\ \left[\mathbf{J}^{\pm|x}\,,\,\mathbf{J}^{\pm|y}\right] &= \epsilon^{xyz}\,J^{\pm|z} \\ \left[\mathbf{J}^{\pm|x}\,,\,\mathbf{J}^{\mp|y}\right] &= 0 \end{aligned} \tag{B.7}$$

Then we introduce a basis of generators for the Lie algebra **so3w4** named as follows:

$$T_I = \{J_x\,,\,W_i\} = \{J_1, J_2, J_3, W_1, W_2, W_3, W_4\} = \{J_x\,,\,W_i\} \tag{B.8}$$

and the only possible commutation relations are the following ones:

$$\begin{aligned}
\left[J^x\,,\,J^y\right] &= \epsilon^{xyz}\,J^z \\
\left[J^x\,,\,W^i\right] &= (\alpha \mathbf{J}^{+|x}_{ij} + \beta \mathbf{J}^{-|x}_{ij})\,W_j \\
\left[W_i\,,\,W_j\right] &= 0
\end{aligned} \tag{B.9}$$

where

$$(\alpha\,,\,\beta) = \begin{cases} (1,0) \text{ or} \\ (1,1) \text{ or} \\ (0,1) \end{cases} \tag{B.10}$$

Indeed the 4-dimensional radical has necessarily to be abelian, since there is no SO(3) invariant three index tensor in the spinor representation, or to say it differently the tensor cube of the $j = \frac{1}{2}$ representation does not contain the singlet.

An explicit representation of the 't Hooft matrices is the following one:

$$\begin{aligned}
J^{+|1} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} \quad ; \quad J^{-|1} &= \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} \\
J^{+|2} &= \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} \quad ; \quad J^{-|2} &= \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} \\
J^{+|3} &= \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \quad ; \quad J^{-|31} &= \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{B.11}$$

and it can be used to perform an explicit calculation of the Ricci form via eq.(4.7). In the three cases provided by eq.s (B.9) and (B.10) we obtain three times the same result, namely:

$$\mathbf{Ric}_{so3w4} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{B.12}$$

This shows that for this algebra the Ricci form is degenerate and has 4 vanishing eigenvalues. It can be discarded as a candidate for weak G_2 holonomy.

B.2 Lie algebras $\mathbb{G}_7$ with Levi subalgebra $\mathbb{L}(\mathbb{G}_7) = \mathbf{so(1,2)}$

These Lie algebras are the semidirect product of the pseudo rotation algebra SO(1, 2) with a four dimensional solvable Lie algebra $\mathcal{S}_4$ which must carry a real representation of SO(1, 2). Here the situation is different from the case of the compact Levi algebra SO(3) since the two dimensional representation of SO(1, 2) can be chosen real and identified with the defining representation of SL(2, $\mathbb{R}$). Hence we have the following subcases, depending on the representation assignments of the $\mathcal{S}_4$ radical:

a) $\mathcal{S}_4 = \mathbf{4}$

b) $\mathcal{S}_4 = \mathbf{2} \oplus \mathbf{2}$

c) $\mathcal{S}_4 = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}$

d) $\mathcal{S}_4 = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$

where $\mathbf{1}$ denotes the singlet representation, $\mathbf{2}$ denotes the $j = \frac{1}{2}$ representation and $\mathbf{4}$ denotes the irreducible $j = \frac{3}{2}$ representation. Let us discuss these four cases separately.

B.2.1 a) The irreducible case a

Here the structure of the Lie algebra is completely fixed. Let

$$\begin{aligned}
\Lambda_1 &= \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \\
\Lambda_2 &= \begin{pmatrix} 0 & \frac{3}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{2} & 0 \end{pmatrix} \\
\Lambda_3 &= \begin{pmatrix} 0 & \frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} & 0 \end{pmatrix}
\end{aligned} \tag{B.13}$$

be the generators of SO(1,2) in the irreducible $j = \frac{3}{2}$ representation which corresponds to the 3-times symmetric tensor product of the fundamental $j = 1$ representation:

$$
\begin{aligned}
\lambda_1 &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\
\lambda_2 &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}
\end{aligned}
\tag{B.14}
$$

Naming $f^x{}_{yz}$ the structure constants of the SO(1,2) simple Lie algebra:

$$
[J_y\,,\,J_z] = f^x{}_{yz}\,J_z \quad ; \quad [\Lambda_y\,,\,\Lambda_z] = f^x{}_{yz}\,\Lambda_z \quad ; \quad [\lambda_y\,,\,\lambda_z] = f^x{}_{yz}\,\lambda_z
\tag{B.15}
$$

the only Lie possible Lie algebra corresponding to this case is given by:

$$
\begin{aligned}
[J_x\,,\,J_y] &= f^x{}_{yz}\,J^z \\
[J_x\,,\,W_i] &= (\Lambda_z)_i^{\ j}\,W_j \\
[W_i\,,\,W_j] &= 0
\end{aligned}
\tag{B.16}
$$

where J_x ($x = 1,2,3$) are the generators of SO(1,2) and W_i ($i = 1,2,3,4$) are the generators spanning the $j = \frac{3}{2}$ representation. We name the above algebra **so12w4**. If we order the seven generators in the following way:

$$
\{J_1, J_2, J_3, W_1, W_2, W_3, W_4\}
\tag{B.17}
$$

we can evaluating the Ricci form by means of the formula (4.7) and we find:

$$
\mathbf{Ric}_{so12w4} = \begin{pmatrix}
-\frac{13}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{B.18}
$$

As we see, also in this case the Ricci form has both positive and negative eigenvalues, so also **w4so12** is ruled out as a candidate for weak G_2 holonomy.

B.2.2 b) The case of two doublets

In this case the Lie algebra is also completely fixed by the choice of the representations. The seven generators are arranged into one triplet J_x ($x = 1,2,3$), corresponding to

the adjoint representation and two doublets W_α and U_α ($\alpha = 1, 2$). The commutation relations are:

$$
\begin{aligned}
[J_x\,,\,J_y] &= f^x{}_{yz}\,J^z \\
[J_x\,,\,W_\alpha] &= (\lambda_z)_\alpha^{\ \beta}\,W_\beta \\
[J_x\,,\,U_\alpha] &= (\lambda_z)_\alpha^{\ \beta}\,U_\beta \\
[W_\alpha\,,\,W_\beta] &= 0 \\
[U_\alpha\,,\,U_\beta] &= 0 \\
[W_\alpha\,,\,U_\beta] &= 0
\end{aligned}
\tag{B.19}
$$

where the matrices λ_x were defined in eq. (B.14) and the SO(1,2) structure constants in (B.15). We name the above algebra **so12w2u2**. Arranging the generators in the following order:

$$\{J_1, J_2, J_3, W_1, W_2, U_1, U_2\} \tag{B.20}$$

The calculation of the Ricci form yields:

$$
\mathbf{Ric}_{so12w2u2} = \begin{pmatrix}
-\frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{5}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{B.21}
$$

We have positive, negative and null eigenvalues. Hence also this algebra is ruled out.

B.2.3 c) The case of one doublet and two singlets.

With these representation assignments there is more than one algebra which is possible. So we have to study a few subcases. Let us name J_x the triplet generators and W_α the doublet ones. The remaining two, we can call Z and U. The following commutation relations are fixed by the representation assignments:

$$
\begin{aligned}
[J_x\,,\,J_y] &= f^x{}_{yz}\,J^z \\
[J_x\,,\,W_\alpha] &= (\lambda_z)_\alpha^{\ \beta}\,W_\beta \\
[J_x\,,\,U] &= 0 \\
[J_x\,,\,Z] &= 0
\end{aligned}
\tag{B.22}
$$

For the remaining commutators we have a bifurcation. We can decide that the doublet generators are not abelian, rather, together with one of the singlet, let us say Z, they form a nilpotent algebra. This is possible because the antisymmetric square of the $j = \frac{1}{2}$ representation contains the singlet. Hence we can write:

$$[W_\alpha\,,\,W_\beta] = \epsilon^{\alpha\beta}\,Z \tag{B.23}$$

If we choose this path then the most general commutation relations with the last generator U are fixed by Jacobi identities and are the following one:

$$\begin{aligned} [U\,,\,W_\alpha] &= q\,W_\alpha \\ [U\,,\,Z] &= 2\,q\,Z \end{aligned} \tag{B.24}$$

The algebra defined by the commutation relations (B.22,B.23,B.24) we name **so12w2uz** and we can calculate its Ricci form by arranging its generators in the following order:

$$\{J_1, J_2, J_3, W_1, W_2, Z, U\} \tag{B.25}$$

The result is:

$$\mathbf{Ric}_{so12w2uz} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} - 2\,q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} - 2\,q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} - 4\,q^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3\,q^2 \end{pmatrix} \tag{B.26}$$

By explicit inspection we see that irrespectively of the value of q, the signature of the Ricci form is indefinite including both positive and negative eigenvalues. Hence also this case has to be discarded.

The other possibility for a Lie algebra with the chosen representation assignments is realized by keeping eq.(B.22) and replacing eq. (B.23) with:

$$[W_\alpha\,,\,W_\beta] = 0 \tag{B.27}$$

At the same time we can replace eq.(B.24) with

$$\begin{aligned} [U\,,\,W_\alpha] &= q_2\,W_\alpha \\ [Z\,,\,W_\alpha] &= q_1\,W_\alpha \\ [U\,,\,Z] &= 0 \end{aligned} \tag{B.28}$$

The algebra defined by eq.s(B.22,B.27,B.28), we name **so12w2q1q2**. We can calculate its Ricci form by arranging its generators in the same order (B.25) as above. The result is given below:

$$\mathbf{Ric}_{so12w2q1q2} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -{q_1}^2 - {q_2}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -{q_1}^2 - {q_2}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -{q_1}^2 & -(q_1\,q_2) \\ 0 & 0 & 0 & 0 & 0 & -(q_1\,q_2) & -{q_2}^2 \end{pmatrix} \tag{B.29}$$

As we see also this Ricci form has indefinite signature. It has both positive and negative eigenvalues and also a null eigenvalue. Hence also this case has to be discarded.

B.2.4 d) The case of four singlets

This case is easily disposed of. If four of the seven generators are singlets, then it means that our algebra is actually the direct product of the Levi subalgebra $SO(1,2)$ which by itself has already an indefinite Ricci form with a completely solvable subalgebra for which the Ricci form has at least one negative eigenvalue as we have proved in the main text. Indeed for direct product algebras the Ricci form is obviously block diagonal. So all such cases are automatically excluded.

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